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# Fractional quantum hall effect in nonuniform magnetic fields

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Investigations of the fractional quantum Hall effect are extended to spatially varying magnetic fields. Approximate single-particle wave functions are proposed and compared with ones obtained by numerical integration. As in the uniform field case, the interacting many-electron system forms an incompressible fluid and has fractionally charged excitations. Field inhomogeneities can trap collective excitations.

## I. INTRODUCTION

Laughlin's wave function<sup>1,2</sup> for the ground state of a system exhibiting the fractional quantum Hall effect (FQHE) has many remarkable features. It describes the ground state at a filling fraction  $\nu = 1/k$ , where  $k$  is an odd integer, as an incompressible quantum fluid, and it accounts for the topological excitations needed to describe the system at a filling fraction slightly away from the one above and permits the study of low-lying collective modes. To date<sup>2,3</sup> theoretical, as well as experimental, studies have limited themselves to the situation where the magnetic field is spatially uniform. This work will address itself to the FQHE in spatially varying, albeit not too violently, magnetic fields. Such a situation provides us with a terrain for studying Laughlin-type wave functions in a somewhat different setting and may lead to interesting experimental situations not obtainable in the uniform field case.

Such field inhomogeneities can be created by placing material exhibiting the FQHE between Type-II superconductors. The magnetic field coming out of vortices present in such superconductors would provide the desired nonuniformities. Vortices of 200–300 Å would have a field of 1–2 T threading through them. Thus for typical situations we could achieve 10–20 % field nonuniformity over regions a few magnetic lengths. In subsequent discussions we shall use the above as a generic situation.<sup>4</sup>

As the many interacting electron wave functions for the FQHE are built up of single particle ones in Sec. II we study the quantum mechanics of a single electron in a nonuniform magnetic field. Although this problem cannot be solved exactly, we will obtain approximate wave functions, which, as we shall show, are quite good as long as the magnetic field does not vary too rapidly; this restriction will become more precise in the course of working out these approximate single-particle wave functions. A discussion on the constraints of many-body variational wave functions and the form of such functions are presented in Sec. III. In Sec. IV this discussion is extended to quasiparticle and quasihole excitations and collective mode excitations. For the former, a careful discussion

of superposition of such excitations has to be made; for the latter, interesting, spatially localized collective modes may occur. The study of these many-body wave functions, both for the ground state and for the excitations, is not on as firm a footing as for the noninteracting system as detailed calculations, using either Monte Carlo or fluid mechanics methods,<sup>1</sup> have not been undertaken.

## II. APPROXIMATE SINGLE-PARTICLE WAVE FUNCTIONS

We have in mind the situation of an electron moving in a spatially nonuniform magnetic field  $B(\mathbf{r})$ , which at large distances approaches a uniform value  $B_\infty$ . Let  $S(\mathbf{r})$  be a two dimensional electrostatic potential caused by a charge distribution  $\rho = B(\mathbf{r})/2\pi$ ; with the asymptotic condition on  $B(\mathbf{r})$  discussed above,

$$S(\mathbf{r}) = \frac{1}{2\pi} \int d^2 r' \ln |\mathbf{r} - \mathbf{r}'| [B(\mathbf{r}') - B_\infty] + \frac{1}{4} B_\infty r^2. \quad (2.1)$$

The vector potential  $\mathbf{A}(\mathbf{r})$  in the Coulomb gauge, is related to  $S(\mathbf{r})$  by

$$\mathbf{A}(\mathbf{r}) = \hat{\mathbf{z}} \times \nabla S(\mathbf{r}). \quad (2.2)$$

The Hamiltonian for this problem is

$$H = \frac{1}{2\mu} \left( [-i\nabla - e\mathbf{A}(\mathbf{r})]^2 - g \frac{\sigma_z}{2} B(\mathbf{r}) \right); \quad (2.3)$$

we have set  $\hbar = c = 1$  and  $g$  is the gyromagnetic ratio of the electron. It is remarkable that for  $g = 2$  we can find the exact lowest Landau level wave functions. The eigenfunctions are

$$\psi_m(\mathbf{r}) = N_m z^m \exp[-eS(\mathbf{r})], \quad (2.4)$$

where  $z = x + iy$ , and  $x, y$  are the Cartesian coordinates of the position vector  $\mathbf{r}$ . The eigenvalue of this level is  $E = 0$ . Note that for magnetic fields that are not axially symmetric, the above is not an eigenfunction of angular momentum, as  $S(\mathbf{r})$  depends on the azimuthal angle.

Even though  $g = 2$  is very close to the true gyromagnetic ratio of the *free* electron, it is way off for the situations of interest to the FQHE; in materials that exhibit the FQHE, the effective mass of the electron is much smaller than the free value as is the effective magnetic moment, making the effective  $g$  quite small.<sup>5</sup> Nevertheless, for magnetic fields varying not too rapidly, we shall show that Eq. (2.4) is a good approximate wave function with approximate energy

$$E_m = \langle e(1 - g/2)B(\mathbf{r})/2\mu \rangle_m. \quad (2.5)$$

The expectation is taken in the state corresponding to the wave function of Eq. (2.4).

Prior to studying the validity of Eq. (2.4) as an approximate wave function, it will be useful to compare it to what we expect the true single-particle wave function to look like. We have in mind the generic case of a uniform magnetic field  $B_\infty$  perturbed in a finite region by some extra flux. Consider first the limiting case of a uniform magnetic field to which is added a field  $\Phi\delta(\mathbf{r})$ . It is easy to check that Eq. (2.4) is exact for all angular momenta  $m$  when the sign of  $\Phi$  is opposite to that of  $B_\infty$ , and for all  $m \geq \Phi/\Phi_0$  in the opposite situation;  $\Phi_0 = 2\pi/e$ , is the quantum flux unit. Thus, we expect that our approximate wave function is good for states that have a small overlap with regions of the perturbing flux. Similarly, states whose wave functions are peaked in a region where the perturbing flux is relatively constant will feel only the perturbing field and be insensitive to  $B_\infty$ ; again the approximate wave function will be good. It is suspect for states whose wave function are large in transition regions.

Numerical studies show that it is good even for the last situations mentioned above. In Table I we present a comparison of results obtained using our approximate wave function to those obtained from numerical integrations;

the gyromagnetic ratio  $g$  was set to zero. The magnetic field was taken to have a sharp discontinuity at a radius  $R_0$ ;  $eB = 1$  for  $r > R_0$  and  $eB = B_0$  for  $r < R_0$ . In accordance with the discussions in Sec. I, we let  $B_0$  differ by 20% from unity and let  $R_0$  vary from a fraction to several magnetic lengths,  $1/eB_\infty$ . Even though this example is extreme, in that the magnetic field has a sharp discontinuity, the expectations discussed above are borne out. For small values of  $R_0$  our approximation becomes better as  $m$  increases, while for larger  $R_0$  it is good for both small and large values of  $m$  and deviates at intermediate values. (Note that for the above choice of magnetic field the wave functions peak for radii of the order of  $m$ .) We should also note that the results are good for all values of  $m$ . The eigenvalues do not deviate by more than 1.5% and the overlap of the approximate and true wave functions is close to one. We shall refer to these states as the “lowest Landau level states.”

### III. MANY-BODY WAVE FUNCTION

We will now construct a variational wave function for the interacting  $N$  electron problem. In the uniform field case,<sup>6</sup> three constraints determined the form of this function. Two of these can be applied to the present situation. (i) The wave function must be totally antisymmetric. (ii) The many body wave function is composed of single particle states in the lowest Landau level. These conditions restrict a Jastrow variational wave function to the form

$$\Psi(\mathbf{r}_1, \dots, \mathbf{r}_N) = \prod_{i < j} [f(z_i) - f(z_j)] \exp \left( -e \sum_i S(\mathbf{r}_i) \right), \quad (3.1)$$

with  $f(z)$  an odd polynomial. The third condition in the uniform field case, namely, that the wave function be an eigenfunction of angular momentum, restricting  $f(z)$  to

TABLE I. Comparison of approximate [Eq. (2.5)] and exact (numerical) single-particle wave functions for various field inhomogeneities and angular momenta.

$m$	$B_0/B_\infty = 0.8$			$B_0/B_\infty = 1.2$		
	$E_{\text{exact}}$	$E_{\text{Eq. (2.5)}}$	Overlap	$E_{\text{exact}}$	$E_{\text{Eq. (2.5)}}$	Overlap
$R_0 = 0.2$						
0	0.999	0.997	0.998	1.010	1.008	0.998
1	0.999	1.000	0.999	0.999	1.001	0.999
2	1.000	1.000	1.000	1.000	1.000	1.000
3	1.000	1.000	1.000	1.000	1.000	1.000
$R_0 = 1.0$						
0	0.927	0.923	0.999	1.089	1.090	0.999
1	0.984	1.000	1.000	1.019	1.024	1.000
2	0.997	0.997	1.000	1.029	1.030	1.000
3	1.000	1.000	1.000	1.000	1.000	1.000
$R_0 = 3.0$						
0	0.806	0.804	1.000	1.200	1.199	1.000
1	0.821	0.818	1.000	1.192	1.193	1.000
2	0.854	0.844	0.999	1.178	1.179	1.000
3	0.894	0.879	0.999	1.153	1.151	0.999

be a monomial, cannot be applied to the present situation. Fermi statistics and the strongly repulsive electron-electron interaction force  $f(z)$  to vanish at  $z = 0$ ; were  $f(z)$  to be more complicated than a monomial it would also vanish for other fixed interparticle separations. Such zeros act as additional vortices of magnetic flux. In the uniform field case, such vortices correspond to quasiparticle excitations and cost energy. We shall assume that also in the present case there are no extra zeros and that  $f(z) = z^k$ , for some odd integer  $k$ . Up to normalization we take as the wave function for the  $N$  electron system

$$\Psi_k(\mathbf{r}_1, \dots, \mathbf{r}_N) = \prod_{i < j} (z_i - z_j)^k \exp \left( -e \sum_i S(\mathbf{r}_i) \right). \quad (3.2)$$

In order to understand the nature of this state we write the square of  $\Psi$  as

$$|\Psi_k|^2 = \exp(-\beta\Phi), \quad (3.3)$$

with

$$\Phi = - \sum_{i < j} 2k^2 \ln |\mathbf{r}_i - \mathbf{r}_j| + 2ke \sum_i S(\mathbf{r}_i) \quad (3.4)$$

and  $\beta = 1/k$ . This is the canonical ensemble probability for a two-dimensional gas of particles of charge  $-k$  in a positive background with density  $\rho_k = eB(\mathbf{r})/2\pi k$ . In order to neutralize this nonuniform positive background, the electrons will distribute themselves with the above density. This state has a local filling fraction  $\nu = 1/k$  in the sense that the electron density

$$\rho(\mathbf{r}) = \nu \frac{B(\mathbf{r})}{\Phi_0}. \quad (3.5)$$

This system forms an incompressible fluid, as for a fixed number of electrons the total flux cannot be changed; the spatial dependence of the magnetic field may be varied as long as the total flux is left unchanged.

## IV. EXCITATIONS

### A. Quasiparticles

If the total number of electrons is such that Eq. (3.5) cannot be maintained for any odd integer filling fraction, quasiholes and/or quasiparticles, corresponding to vortices of extra magnetic flux, will appear. The wave function for a quasihole located at  $z_0$  is taken to be

$$\begin{aligned} & \Psi_k^{(-)}(\mathbf{r}_1, \dots, \mathbf{r}_N; z_0) \\ &= \exp \left( -e \sum_i S(\mathbf{r}_i) \right) \prod_i (z_i - z_0) \prod_{i < j} (z_i - z_j)^k, \end{aligned} \quad (4.1)$$

while that of the quasiparticle is written as

$$\Psi_k^{(+)}(\mathbf{r}_1, \dots, \mathbf{r}_N; z_0)$$

$$= \exp \left( -e \sum_i S(\mathbf{r}_i) \right) \mathcal{P} \prod_i (z_i^* - z_0^*) \prod_{i < j} (z_i - z_j)^k, \quad (4.2)$$

where  $\mathcal{P}$  projects what follows onto the lowest Landau level. For a uniform magnetic field, this projection is accomplished by replacing  $z_i^*$  by  $2l_0^2 \partial/\partial z_i$  with  $l_0^2 = 1/eB$ . For the nonuniform case no such simple expression exists. Instead, we have

$$\mathcal{P} z^* z^m = \left( \frac{N_{m-1}}{N_m} \right)^2 z^{m-1}. \quad (4.3)$$

$N_m$  is the normalization of the single-particle wave functions as given in Eq. (2.4). As in the uniform field case, we may show that these excitations have charge  $e^* = \mp e/k$ .<sup>1,7,8</sup> This is most easily seen following the techniques of Ref. 8. The change of phase  $\gamma$  of the wave function as we adiabatically move a quasiparticle around a loop is

$$\gamma = -2\pi \langle n \rangle, \quad (4.4)$$

where  $\langle n \rangle$  is the number of electrons in the area bounded by the loop. Using Eq. (3.5) we obtain

$$\gamma = -2\pi\Phi/(k\Phi_0), \quad (4.5)$$

where  $\Phi$  is the flux traversing the area bounded by the loop. This results in a quasiparticle charge discussed above. If there is a second quasiparticle present inside the loop, then an extra phase,  $\delta\gamma = 2\pi/k$  is added to  $\gamma$ ; this is interpreted by assigning fractional statistics to the quasiparticles.

For both the uniform and nonuniform field case these quasiparticle states are, for different vortex positions,  $z_0$ , not orthogonal. In the uniform case the overlap decreases exponentially as  $\exp(-|z_0 - z'_0|^2/2l_0^2)$ , and approaches a  $\delta$  function for small  $l_0$ 's. For any magnetic fields let

$$\begin{aligned} \langle \psi_k(z_0) | \psi_k(z'_0) \rangle &= n(z_0, z'_0), \\ \langle \psi_k(z_0) | H | \psi_k(z'_0) \rangle &= h(z_0, z'_0). \end{aligned} \quad (4.6)$$

Orthogonal quasiparticle states are obtained as linear superposition of Eq. (4.1) or Eq. (4.2),

$$\chi(\mathbf{r}_1, \dots, \mathbf{r}_N) = \int d^2 z_0 \chi(z_0) \psi_k(\mathbf{r}_1, \dots, \mathbf{r}_N; z_0), \quad (4.7)$$

where  $\chi(z_0)$  satisfies

$$\int d^2 z'_0 [h(z_0, z'_0) - \epsilon n(z_0, z'_0) \chi(z'_0)] = 0. \quad (4.8)$$

In the uniform field case both  $n$  and  $h$  are functions of the difference of  $z_0$  and  $z'_0$  and plane waves solve Eq. (4.8); we find momentum states with  $\epsilon(\mathbf{k}) = \tilde{h}(\mathbf{k})/\tilde{n}(\mathbf{k})$ ,  $\tilde{h}$  and  $\tilde{n}$  are Fourier transforms of  $h$  and  $n$ , respectively. In the local limit

$$\begin{aligned} n(z_0, z'_0) &= \delta(z_0 - z'_0), \\ h(z_0, z'_0) &= C^\pm e^2 / l_0 \delta(z_0 - z'_0), \end{aligned} \quad (4.9)$$

$\epsilon$  is independent of  $\mathbf{k}$ , and Eqs. (4.1) and (4.2) are good descriptions of these excitations. In this case the energies are  $E^\pm = C^\pm e^2 / l_0$ ;  $C^\pm$  are constants, independent of  $z_0$ , whose evaluation is quite involved.<sup>3</sup> To date, only this local limit has been considered. Independently of specific spatial dependence of  $B(\mathbf{r})$ , it is only in such a limit that we can make some general statements on the energies of these excitations. The wave functions, Eq. (4.1) and Eq. (4.2), differ significantly from the ground-state wave function only in the vicinity of  $z_0$  and thus are primarily sensitive to the magnetic field at that point. The magnetic field enters the variational energy only through the dependence of the magnetic length on it. In this limit  $l_0$  in Eq. (4.9) will be replaced by the local magnetic length,  $l_0(\mathbf{r})^2 = 1/eB(\mathbf{r})$  and the energy will be  $\mathbf{r}$  dependent,

$$E^\pm(\mathbf{r}) = C^\pm \frac{e^2}{l_0(\mathbf{r})}. \quad (4.10)$$

These excitations will experience a force pushing them to regions of smaller magnetic fields. As discussed earlier, we may vary the  $\mathbf{r}$  dependence of the field as long as we keep the total flux constant; this will cause such excitations to move.

### B. Collective modes

In the uniform field case, collective excitations, with energy  $\Delta$  and wave vector  $\mathbf{k}$ , exist. The dispersion relation has the form

$$\Delta(k) = \frac{e^2}{l_0} F(k l_0). \quad (4.11)$$

The function  $F$  is found<sup>9</sup> to have a “roton” minimum at a nonzero value of  $\mathbf{k}$  occurring at  $k l_0$  of the order of 1. The results of Ref. 9 show that near this minimum Eq. (4.11) may be approximated as

$$\Delta(k) = \frac{e^2}{l_0} [a^2 (k l_0 - b)^2 + d^2], \quad (4.12)$$

where  $\nu = \frac{1}{3}$ ,  $a = 1.25$ ,  $b = 1.28$ , and  $d = 0.87$ . As such a collective mode travels through a region of a slowly varying magnetic field its dispersion relation will be modified in that  $l_0$  will be replaced by  $l_0(\mathbf{r})$ , analogous to position dependence of an index of refraction. In regions with a magnetic field lower than that of its surroundings, discrete, trapped collective bound states may appear below the roton minimum. We may use a WKB method to find the energies of such localized, “bound” states. As an example consider the situation where a constant magnetic field,  $B$ , drops to a fraction,  $f$ , of its value in a region of spatial extent  $R$ . In this region the dispersion relation for a collective excitation is the same as that of Eq. (4.12), with the magnetic length  $l_0$  changed to the local one,  $l_0(\mathbf{r}) = l_0 / \sqrt{f}$ . The WKB bound state condition is  $kR = \pi$ . For  $R \sim 3.5 l_0$ , we find  $\Delta = e^2 \sqrt{f} d^2 / l_0$ . Specifically, if we take  $B = 12$  T,  $f = 0.8$ , and  $R = 250$  Å a bound state with energy of 10% below the uniform field roton minimum will exist.

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